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A weak energy stationary action principle for quantum state evolution

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Abstract

It is shown that the actual paths in Hilbert space followed by a finite set of $n \geqslant 2$ quantum states evolving between initial and final end point configurations are such that an associated weak energy functional defined by Pancharatnam phases and state separation distances in projective Hilbert space determined by the generalized Fubini-Study metric is stationary for all variations of these phases, separations and time which vanish at the end points. Noether's theorem is used to identify two weak energy conservation laws which are shown to be the analogues of the momentum and energy conservation laws of Langrangian mechanics.

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1. Introduction

Variational principles are well known in classical mechanics for their unifying qualities and philosophical appeal. For example, rather than independently addressing the state of a mechanical system at each instant of time during its motion, *Hamilton's principle* treats the motion as a whole in a single statement which asserts that if the initial and final configurations of a system are prescribed, then the system's motion occurs in a manner such that the associated definite time integral of the Lagrangian function is stationary for arbitrary variations of the system's configuration. This principle contains the mechanics of holonomic systems with forces derivable from potentials and—when combined with *Noether's theorem*—provides powerful conservation laws. Quantum mechanical variational principles have also been developed by Feynman and Hibbs [1] and Schwinger [2].

The notion of the 'weak value' of a quantum mechanical observable was introduced by Aharonov *et al* [3–5] over a decade ago. This value is the statistical result of a standard measurement procedure upon a pre-selected and post-selected ensemble of quantum systems when the interaction between the measuring apparatus and the system is sufficiently weak.

Since then, weak values have been discussed in a variety of contexts including quantum stochastic processes [6], quantum non-locality [7], conditional probabilities and the tunnelling time controversy [8, 9], physical elements of reality [10], quantum random walks [11], arrival time probability distributions [12], quantum communication protocols [13, 14], counterfactual reasoning [15] and quantum trajectory theory [16].

Recently an intrinsic weak energy has been observed experimentally [17] and studied from a theoretical perspective [18]. The purpose of this paper is to extend this theory by showing that the actual paths in Hilbert space followed by a finite set of $n \geqslant 2$ quantum states evolving between initial and final end point configurations are such that an associated weak energy functional defined by Pancharatnam phases and state separations in projective Hilbert space determined by the generalized Fubini-Study metric is stationary for all variations of these phases, separations and time which vanish at the end points. In addition, Noether's theorem is used to identify two weak energy conservation laws and they are shown to be analogues of the momentum and energy conservation laws of Lagrangian mechanics.

2. Weak energy

Consider a pair $|\psi_j(t)\rangle$ and $|\psi_k(t)\rangle$ of normalized quantum states evolving in a Hilbert space $\mathcal{H}$. Let $\mathcal{P}$ be the associated projective space consisting of all the rays of $\mathcal{H}$ (recall that a ray is an equivalence class $[\psi]$ of states $|\psi\rangle$ in $\mathcal{H}$ which differ only in phase) with $\Pi : \mathcal{H} \to \mathcal{P}$ the induced projection map such that $|\psi\rangle \longmapsto [\psi]$. The weak energy $W_{j,k}(t)$ associated with this pair of states is the complex-valued quantity defined by [17, 18]

$$W_{j,k}(t) \equiv \frac{\langle\psi_j(t)|\hat{H}_j - \hat{H}_k|\psi_k(t)\rangle}{\langle\psi_j(t)|\psi_k(t)\rangle} = \operatorname{Re} W_{j,k}(t) + \mathrm{i} \operatorname{Im} W_{j,k}(t) \tag{1}$$

where $\langle\psi_j(t)|\psi_k(t)\rangle \neq 0$ with

$$\mathrm{i}\hbar\frac{\mathrm{d}|\psi_k(t)\rangle}{\mathrm{d}t} = \hat{H}_k|\psi_k(t)\rangle \qquad \text{and} \qquad \mathrm{i}\hbar\frac{\mathrm{d}\langle\psi_j(t)|}{\mathrm{d}t} = -\langle\psi_j(t)|\hat{H}_j.$$

It is shown in [18] that

$$\operatorname{Re} W_{j,k}(t) = \hbar\left(\frac{\mathrm{d}\chi_{j,k}(t)}{\mathrm{d}t}\right) \equiv \hbar\dot{\chi}_{j,k}(t)$$

and

$$\operatorname{Im} W_{j,k}(t) = \hbar\left\{\frac{s_{j,k}(t)}{4 - s_{j,k}^2(t)}\right\}\left(\frac{\mathrm{d}s_{j,k}(t)}{\mathrm{d}t}\right) \equiv \hbar\left\{\frac{s_{j,k}(t)}{4 - s_{j,k}^2(t)}\right\}\dot{s}_{j,k}(t).$$

Here, $\chi_{j,k}(t)$ is the Pancharatnam phase [19] at time t defined by

$$\mathrm{e}^{\mathrm{i}\chi_{j,k}(t)} = \frac{\langle\psi_j(t)|\psi_k(t)\rangle}{|\langle\psi_j(t)|\psi_k(t)\rangle|} \tag{2}$$

and is the phase difference between $|\psi_k(t)\rangle$ and $|\varphi(t)\rangle$, where $|\varphi(t)\rangle$ is the state contained in the equivalence class $[\psi_k(t)]$ obtained by parallel transporting $|\psi_j(t)\rangle$ along the unique path in $\mathcal{H}$ that is the pre-image under the projection map Π of the shortest geodesic joining $[\psi_j(t)]$ and $[\psi_k(t)]$ in $\mathcal{P}$ [20]. The function $s_{j,k}(t)$ is the distance separating $[\psi_j(t)]$ and $[\psi_k(t)]$ in $\mathcal{P}$ at time t given by the generalized Fubini-Study metric defined by [21, 22]

$$s_{j,k}^2(t) \equiv 4(1 - |\langle\psi_j(t)|\psi_k(t)\rangle|^2). \tag{3}$$

3. The weak energy stationary action principle

Let $J = \{1, 2, \ldots, n\}$, $n \geqslant 2$, index a set $\mathcal{S}(t) = \{|\psi_j(t)\rangle : j \in J\}$ of quantum states evolving in $\mathcal{H}$ during a time interval $[t_0, t_1]$ and define the *weak energy action* I to be the functional

$$I \equiv \int_{t_0}^{t_1} \sum_{j<k\in J} \mathcal{L}(s_{j,k}(t); \dot{\chi}_{j,k}(t), \dot{s}_{j,k}(t))\,\mathrm{d}t = \sum_{j<k\in J} \int_{t_0}^{t_1} \mathcal{L}(s_{j,k}(t); \dot{\chi}_{j,k}(t), \dot{s}_{j,k}(t))\,\mathrm{d}t$$

where

$$\mathcal{L}(s_{j,k}(t); \dot{\chi}_{j,k}(t), \dot{s}_{j,k}(t)) \equiv \hbar\dot{\chi}_{j,k}(t) + \mathrm{i}\hbar \left\{ \frac{s_{j,k}(t)}{4 - s_{j,k}^2(t)} \right\} \dot{s}_{j,k}(t) = W_{j,k}(t). \tag{4}$$

The purpose of this section is to show that *the actual paths followed in $\mathcal{H}$ by the states in $\mathcal{S}(t)$ between the end point configurations $\mathcal{S}(t_0)$ and $\mathcal{S}(t_1)$ at times $t_0 < t_1$ are such that the weak energy action I is stationary for all variations in $\chi_{j,k}$, $s_{j,k}$ and time which vanish at the end points*. We call this assertion the *weak energy stationary action principle (WESAP)*.

In order to show this, consider the general transformations parametrized by the small number ϵ and given by

$$\chi'_{j,k}(t) = \chi_{j,k}(t) + \epsilon\alpha_{j,k}[t] \tag{5}$$

$$s'_{j,k}(t) = s_{j,k}(t) + \epsilon\beta_{j,k}[t] \tag{6}$$

and

$$t' = t + \epsilon\gamma_{j,k}[t] \tag{7}$$

with

$$x_{j,k}[t] \equiv x_{j,k}(t; \chi_{1,2}, \ldots, \chi_{n-1,n}, s_{1,2}, \ldots, s_{n-1,n}; \dot{\chi}_{1,2}, \ldots, \dot{\chi}_{n-1,n}, \dot{s}_{1,2}, \ldots, \dot{s}_{n-1,n})$$
$$x \in \{\alpha, \beta, \gamma\}$$

(the time dependence of $\chi_{i,l}$, $s_{i,l}$ and their rates of change are suppressed for notational brevity). Use equation (7) to write t in terms of t' to first order in ϵ as $t = t' - \epsilon\gamma_{j,k}[t']$ and form the quantity

$$I(\epsilon) = \sum_{j<k\in J} \int_{t'_0}^{t'_1} \mathcal{L}(s^\sharp_{j,k}(t'); \dot{\chi}^\sharp_{j,k}(t'), \dot{s}^\sharp_{j,k}(t'))\,\mathrm{d}t' \tag{8}$$

where

$$\begin{aligned} t'_i &= t_i + \epsilon\gamma_{j,k}[t_i] \qquad i = 0, 1 \\ \chi^\sharp_{j,k}(t') &\equiv \chi'_{j,k}(t' - \epsilon\gamma_{j,k}[t']) \end{aligned} \tag{9}$$

and

$$s^\sharp_{j,k}(t') \equiv s'_{j,k}(t' - \epsilon\gamma_{j,k}[t']) \tag{10}$$

with $\dot{\chi}^\sharp_{j,k}(t')$ and $\dot{s}^\sharp_{j,k}(t')$ defined accordingly with respect to t'.

Expanding equations (9) and (10) through first order in ϵ yields

$$\chi^\sharp_{j,k}(t') = \chi_{j,k}(t') + \epsilon\eta_{j,k}[t']$$

and

$$s^\sharp_{j,k}(t') = s_{j,k}(t') + \epsilon\sigma_{j,k}[t']$$

where

$$\eta_{j,k}[t'] = \alpha_{j,k}[t'] - \frac{\partial \chi_{j,k}}{\partial t'}\gamma_{j,k}[t']$$

and

$$\sigma_{j,k}[t'] = \beta_{j,k}[t'] - \frac{\partial s_{j,k}}{\partial t'}\gamma_{j,k}[t'].$$

Substitution of these expanded expressions and their time derivatives into equation (8) gives

$$I(\epsilon) = \sum_{j<k\in J} \int_{t_0'}^{t_1'} \mathcal{L}(s_{j,k}(t') + \epsilon\sigma_{j,k}[t'];\ \dot{\chi}_{j,k}(t') + \epsilon\dot{\eta}_{j,k}[t'], \dot{s}_{j,k}(t') + \epsilon\dot{\sigma}_{j,k}[t'])\,\mathrm{d}t'$$

which—upon expansion through first order in ϵ—becomes

$$I(\epsilon) = \sum_{j<k\in J} \int_{t_0'}^{t_1'} \left\{\mathcal{L}(s_{j,k}(t');\ \dot{\chi}_{j,k}(t'), \dot{s}_{j,k}(t')) + \epsilon\left[\sigma_{j,k}[t']\frac{\partial\mathcal{L}}{\partial s_{j,k}} + \dot{\eta}_{j,k}[t']\frac{\partial\mathcal{L}}{\partial\dot{\chi}_{j,k}} + \dot{\sigma}_{j,k}[t']\frac{\partial\mathcal{L}}{\partial\dot{s}_{j,k}}\right]\right\}\mathrm{d}t'. \tag{11}$$

Now use the fact that $\int_{t_0'}^{t_1'}\mathrm{d}t' = \int_{t_0}^{t_1}\mathrm{d}t' - \int_{t_0}^{t_0'}\mathrm{d}t' + \int_{t_1}^{t_1'}\mathrm{d}t'$ to rewrite the last equation to first order in ϵ as

$$I(\epsilon) = \sum_{j<k\in J}\left\{\begin{array}{c} \int_{t_0}^{t_1}\mathcal{L}(s_{j,k}(t');\ \dot{\chi}_{j,k}(t'), \dot{s}_{j,k}(t'))\,\mathrm{d}t' \\ +\epsilon\int_{t_0}^{t_1}\left[\sigma_{j,k}[t']\frac{\partial\mathcal{L}}{\partial s_{j,k}} + \dot{\eta}_{j,k}[t']\frac{\partial\mathcal{L}}{\partial\dot{\chi}_{j,k}} + \dot{\sigma}_{j,k}[t']\frac{\partial\mathcal{L}}{\partial\dot{s}_{j,k}}\right]\mathrm{d}t' \\ -\int_{t_0}^{t_0'}\mathcal{L}(s_{j,k}(t');\ \dot{\chi}_{j,k}(t'), \dot{s}_{j,k}(t'))\,\mathrm{d}t' + \int_{t_1}^{t_1'}\mathcal{L}(s_{j,k}(t');\ \dot{\chi}_{j,k}(t'), \dot{s}_{j,k}(t'))\,\mathrm{d}t' \end{array}\right\}. \tag{12}$$

Since the integration ranges for $\int_{t_i}^{t_i'}\mathrm{d}t'$, $i = 0, 1$, are proportional to ϵ, then (i) the integrals with these integration limits of the term in brackets in expression (11) do not appear in the last equation because they are of order ϵ^2; and (ii) the last two integrals in the last equation can be written as

$$\epsilon\gamma_{j,k}[t_i]\mathcal{L}(s_{j,k}(t_i);\ \dot{\chi}_{j,k}(t_i), \dot{s}_{j,k}(t_i)) \qquad i = 0, 1.$$

Substituting this into equation (12) and integrating by parts the last two terms of the integrand in brackets in this equation yield—upon rearrangement—the difference quotient

$$\frac{I(\epsilon) - I(0)}{\epsilon} = \sum_{j<k\in J}\left\{\begin{array}{c} \int_{t_0}^{t_1}\left\{\sigma_{j,k}[t']\left[\frac{\partial\mathcal{L}}{\partial s_{j,k}} - \frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{\partial\mathcal{L}}{\partial\dot{s}_{j,k}}\right)\right] - \eta_{j,k}[t']\frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{\partial\mathcal{L}}{\partial\dot{\chi}_{j,k}}\right)\right\}\mathrm{d}t' \\ +\left[\eta_{j,k}[t']\left(\frac{\partial\mathcal{L}}{\partial\dot{\chi}_{j,k}}\right) + \sigma_{j,k}[t']\left(\frac{\partial\mathcal{L}}{\partial\dot{s}_{j,k}}\right) + \gamma_{j,k}[t']\mathcal{L}(s_{j,k}(t');\ \dot{\chi}_{j,k}(t'), \dot{s}_{j,k}(t'))\right]_{t_0}^{t_1} \end{array}\right\} \tag{13}$$

where

$$I(0) = I = \sum_{j<k\in J}\int_{t_0}^{t_1}\mathcal{L}(s_{j,k}(t');\ \dot{\chi}_{j,k}(t'), \dot{s}_{j,k}(t'))\,\mathrm{d}t'.$$

It is readily verified from equation (4) that

$$\frac{\partial\mathcal{L}}{\partial s_{j,k}} - \frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{\partial\mathcal{L}}{\partial\dot{s}_{j,k}}\right) = 0 = \frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{\partial\mathcal{L}}{\partial\dot{\chi}_{j,k}}\right) \tag{14}$$

so that the integrand on the right-hand side (rhs) of equation (13) vanishes and

$$\frac{I(\epsilon)-I(0)}{\epsilon} = \sum_{j<k\in J}\left[\eta_{j,k}[t']\left(\frac{\partial\mathcal{L}}{\partial\dot{\chi}_{j,k}}\right)+\sigma_{j,k}[t']\left(\frac{\partial\mathcal{L}}{\partial\dot{s}_{j,k}}\right)\right. \\ \left. +\gamma_{j,k}[t']\mathcal{L}(s_{j,k}(t');\dot{\chi}_{j,k}(t'),\dot{s}_{j,k}(t'))\right]_{t_0}^{t_1}.$$

If it is required that all $\chi_{j,k}, s_{j,k}$ and time variations vanish at the end points, then $\eta_{j,k}[t_i] = \sigma_{j,k}[t_i] = \gamma_{j,k}[t_i] = 0$ for $i = 0, 1$ and the value of the rhs of the last equation is zero. Recognizing that this implies

$$\frac{\mathrm{d}I}{\mathrm{d}\epsilon} = \lim_{\epsilon\to 0}\frac{I(\epsilon)-I(0)}{\epsilon} = 0$$

leads to the conclusion that I is stationary.

4. Symmetries and conservation laws

Let us agree to call the $U(1)$ transformations $|\psi_j\rangle \to \mathrm{e}^{\mathrm{i}\theta_j}|\psi_j\rangle$, where θ_j is a constant phase angle, '*global transformations*' and observe that their application to equation (2) yields

$$\mathrm{e}^{\mathrm{i}\chi'_{j,k}} = \frac{\langle\psi_j|\psi_k\rangle}{|\langle\psi_j|\psi_k\rangle|}\,\mathrm{e}^{\mathrm{i}(\theta_k-\theta_j)} = \mathrm{e}^{\mathrm{i}[\chi_{j,k}+(\theta_k-\theta_j)]} \Rightarrow \chi'_{j,k} = \chi_{j,k}+(\theta_k-\theta_j).$$

In addition to $\chi'_{j,k}(t) = \chi_{j,k}(t)+(\theta_k-\theta_j)$, it is also found that $s'_{j,k}(t) = s_{j,k}(t)$ when global transformations are applied to equation (3) (i.e. $U(1)$ is an isometry group for $\mathcal{P}$ when distance is determined by the Fubini-Study metric). Then, since $\dot{\chi}'_{j,k}(t) = \dot{\chi}_{j,k}(t)$, it is readily determined that

$$\mathcal{L}(s'_{j,k}(t);\dot{\chi}'_{j,k}(t),\dot{s}'_{j,k}(t)) = \mathcal{L}(s_{j,k}(t);\dot{\chi}_{j,k}(t),\dot{s}_{j,k}(t))$$

(as required by equation (1)) and

$$\int_{t_0}^{t_1}\sum_{j<k\in J}\mathcal{L}(s'_{j,k}(t);\dot{\chi}'_{j,k}(t),\dot{s}'_{j,k}(t))\,\mathrm{d}t = \int_{t_0}^{t_1}\sum_{j<k\in J}\mathcal{L}(s_{j,k}(t);\dot{\chi}_{j,k}(t),\dot{s}_{j,k}(t))\,\mathrm{d}t$$

so that both $\mathcal{L}$ and I are unchanged under global transformations (i.e. they are '*globally invariant*') and consequently exhibit a '*global symmetry*'. Thus, the action I remains stationary under global transformations and the *WESAP* is valid even if each state in $\mathcal{S}(t)$ is multiplied by a constant phase factor, i.e. when $\mathcal{S}(t) = \{\mathrm{e}^{\mathrm{i}\theta_j}|\psi_j(t)\rangle : j \in J\}$. It may be concluded that this global symmetry leads to a somewhat more 'generalized' *WESAP* which is obtained by amending it's original statement to include the phase '$\cdots$, *even if each state in* $\mathcal{S}(t)$ *is multiplied by a constant phase factor*'.

A physical consequence of this symmetry is the global invariance of the time translation equation for correlation amplitudes given by [18]

$$\langle\psi_j(t_1)|\psi_k(t_1)\rangle = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1}\mathcal{L}(s_{j,k}(t);\dot{\chi}_{j,k}(t),\dot{s}_{j,k}(t))\,\mathrm{d}t}\langle\psi_j(t_0)|\psi_k(t_0)\rangle. \tag{15}$$

This equation clearly remains unchanged under global transformations because of the associated global invariance of $\mathcal{L}$. In addition, the time translation equation for correlation probability given by

$$|\langle\psi_j(t_1)|\psi_k(t_1)\rangle|^2 = \mathrm{e}^{-2\int_{t_0}^{t_1}\left\{\frac{s_{j,k}(t)}{4-s^2_{j,k}(t)}\right\}\dot{s}_{j,k}(t)\,\mathrm{d}t}|\langle\psi_j(t_0)|\psi_k(t_0)\rangle|^2 \tag{16}$$

and which generalizes the temporal persistence of state normalization (because—from equation (1)—when $j = k$, then $s_{j,k}(t) = 0$) is also globally invariant because of the associated global invariance of $s_{j,k}$.

Noether's theorem [23] can be used to determine the conservation law associated with the $U(1)$ global symmetry discussed above. Although the law is tautological for this case, it has an interesting conservation of 'momentum' interpretation. Observe that $\mathcal{L}$ and I are invariant under infinitesimal translations of $\chi_{j,k}$, i.e. when $\alpha_{j,k}[t] = 1$ and $\beta_{j,k}[t] = \gamma_{j,k}[t] = 0$ in transformations (5)–(7). It then follows from Noether's theorem that

$$\sum_{j<k\in J} \left(\frac{\partial \mathcal{L}}{\partial \dot{\chi}_{j,k}} \right) = \hbar \frac{n(n-1)}{2} = \text{constant}, \tag{17}$$

where use has been made of the fact that $\frac{\partial \mathcal{L}}{\partial \dot{\chi}_{j,k}} = \hbar$.

Now note that since $\mathcal{L}$ is independent of $\chi_{j,k}$

$$\frac{\mathrm{d}}{\mathrm{d}t'} \left(\frac{\partial \mathcal{L}}{\partial \dot{\chi}_{j,k}} \right) = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \chi_{j,k}} - \frac{\mathrm{d}}{\mathrm{d}t'} \left(\frac{\partial \mathcal{L}}{\partial \dot{\chi}_{j,k}} \right) = 0$$

so that the conditions specified by equation (14) show that $\mathcal{L}$ satisfies the Euler–Lagrange equations for each $s_{j,k}$ and $\chi_{j,k}$. Consequently, if $\mathcal{L}$ is thought of as a Lagrangian function with generalized 'coordinates' $s_{j,k}$ and $\chi_{j,k}$, then the associated generalized conjugate '*Fubini-Study momentum*' $p_{s_{j,k}}$ and the '*Pancharatnam momentum*' $p_{\chi_{j,k}}$ are given by

$$p_{s_{j,k}} = \frac{\partial \mathcal{L}}{\partial \dot{s}_{j,k}} = \mathrm{i}\hbar \left(\frac{s_{j,k}}{4 - s_{j,k}^2} \right) \tag{18}$$

and

$$p_{\chi_{j,k}} = \frac{\partial \mathcal{L}}{\partial \dot{\chi}_{j,k}} = \hbar \tag{19}$$

respectively. It is clear from equation (19) that the momentum $p_{\chi_{j,k}}$ is a conserved quantity so that equation (17) may be interpreted as the '*law of conservation of Pancharatnam momentum*' given by

$$\sum_{j<k\in J} p_{\chi_{j,k}} = \hbar \frac{n(n-1)}{2}.$$

Thus, the (total) Pancharatnam momentum is a constant of the motion for the evolving states in set $\mathcal{S}(t)$. Note that from this perspective, this conservation law is a consequence of the fact that each $\chi_{j,k}$ is an '*ignorable coordinate*'.

Since $\mathcal{L}$ does not depend upon t explicitly, $\mathcal{L}$ and I also exhibit a time translation symmetry, i.e. they are invariant under the infinitesimal time transformations (5)–(7) when $\alpha_{j,k}[t] = \beta_{j,k}[t] = 0$ and $\gamma_{j,k}[t] = 1$. Straightforward application of Noether's theorem yields the conservation law

$$\sum_{j<k\in J} \left\{ \left(\frac{\partial \mathcal{L}}{\partial \dot{\chi}_{j,k}} \right) \dot{\chi}_{j,k}(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{s}_{j,k}} \right) \dot{s}_{j,k}(t) - \mathcal{L}(s_{j,k}(t); \dot{\chi}_{j,k}(t), \dot{s}_{j,k}(t)) \right\} = \text{constant}. \tag{20}$$

Since $\left(\frac{\partial \mathcal{L}}{\partial \dot{\chi}_{j,k}}\right) = \hbar$ and $\left(\frac{\partial \mathcal{L}}{\partial \dot{s}_{j,k}}\right) = \mathrm{i}\hbar\left(\frac{s_{j,k}}{4 - s_{j,k}^2}\right)$, then

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{\chi}_{j,k}} \right) \dot{\chi}_{j,k}(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{s}_{j,k}} \right) \dot{s}_{j,k}(t) = \mathcal{L}(s_{j,k}(t); \dot{\chi}_{j,k}(t), \dot{s}_{j,k}(t))$$

so that constant $= 0$.

This conservation law also has an interesting interpretation if $\mathcal{L}$ is again thought of as a Lagrangian function. In particular, equations (18) and (19) may be used to write the left-hand side of equation (20) as

$$\mathcal{E} \equiv \sum_{j<k\in J} \{ [p_{\chi_{j,k}} \dot{\chi}_{j,k} + p_{s_{j,k}} \dot{s}_{j,k}] - \mathcal{L}(s_{j,k}; \dot{\chi}_{j,k}, \dot{s}_{j,k}) \}$$

so that $\mathcal{E}$ may be interpreted as the '*Jacobi integral*' of the motion and equation (20) is the '*energy conservation law*'

$$\mathcal{E} = 0.$$

Thus, the vanishing 'Jacobi integral' is a constant of the motion for the evolving states in set $\mathcal{S}(t)$.

5. Closing remarks

Like Hamilton's principle, the *WESAP* is a compact holistic statement which describes the motion of an ensemble of quantum mechanical states in terms of the stationary value of its (weak) energy action integral. Whereas the Lagrangian function is the action integral's integrand for Hamilton's principle, that for the *WESAP* is a weak energy functional defined by Pancharatnam phases (i.e. phase differences within rays of states) and Fubini-Study metric distances (i.e. distances between rays) that are associated with ensemble pairs. Consequently, the Hilbert space evolution of an ensemble of $n \geqslant 2$ quantum states can now be understood from an alternative geometric perspective as the motion of a 'point' along a 'curve' in an $n(n-1)$-dimensional configuration space with these phases and metric distances as the generalized coordinates of the space. The *WESAP* requires that the configuration space curve followed by this point is one for which the weak energy action integral is stationary (indeed, the state solutions to the system's Schrödinger equations define this curve). Because fewer quantities are required when $n < m + 1$, this configuration space description of the evolution of an ensemble of n states in m-dimensional Hilbert space can be more efficient than that obtained using the temporal behaviour of the associated nm basis state probability amplitudes (of course, the phases and metric distances which define the required evolutionary curve in configuration space depend upon these amplitudes because they are defined in terms of inner products of states).

An interesting feature of the weak energy functional $\mathcal{L}(s_{j,k}(t);\, \dot{\chi}_{j,k}(t), \dot{s}_{j,k}(t))$ for a pair of states is that it is a solution to the Euler–Lagrange equations for $s_{j,k}$ and $\chi_{j,k}$. In direct analogy with Hamilton's principle, $\mathcal{L}$ may then be thought of as a Lagrangian function. This—when combined with its intrinsic global $U(1)$ and time translation symmetries—yields quasi momentum and energy conservation laws that are associated with a set of $n \geqslant 2$ evolving quantum states. The invariance under global $U(1)$ transformations of time translations of both correlation and probability amplitudes (equations (15) and (16)) are two important physical consequences of this $U(1)$ symmetry.

Also noteworthy is the observation that these momentum and energy conservation laws provide—in a manner completely consistent with equations (15) and (16)—a formal mechanism for establishing relationships between correlation amplitude and probability time evolution profiles for pairs of states and evolutionary profiles for the weak energy. As an example of this, consider the case where $\mathcal{L}$ is a constant of the motion. Then the energy conservation law $\mathcal{E} = 0$ implies that

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[p_{\chi_{j,k}}\dot{\chi}_{j,k} + p_{s_{j,k}}\dot{s}_{j,k}] = 0.$$

This requires that:

1. $p_{\chi_{j,k}}\dot{\chi}_{j,k} = \textit{real constant} \Rightarrow \dot{\chi}_{j,k} = a_{j,k} = \textit{real constant}$ because $p_{\chi_{j,k}} = \hbar$ (conservation of Pancharatnam momentum) $\Rightarrow$

$$\chi_{j,k}(t) = a_{j,k}t + b_{j,k} \tag{21}$$

where $b_{j,k} = \textit{real constant}$; and

2. $p_{s_{j,k}}\dot{s}_{j,k} = imaginary\ constant \Rightarrow \left(\frac{s_{j,k}}{4-s_{j,k}^2}\right)\dot{s}_{j,k} = \alpha_{j,k} = real\ constant$

$$\Rightarrow \int \left(\frac{s_{j,k}}{4 - s_{j,k}^2}\right) \mathrm{d}s_{j,k} = \alpha_{j,k} \int \mathrm{d}t \Rightarrow s_{j,k}^2 = 4(1 - \mathrm{e}^{-2(\alpha_{j,k}t+\beta_{j,k})})$$

where $\beta_{j,k} = real\ constant \Rightarrow$ from equation (3) that

$$|\langle\psi_j(t)|\psi_k(t)\rangle|^2 = \mathrm{e}^{-2(\alpha_{j,k}t+\beta_{j,k})}. \tag{22}$$

Thus, the correlation probability decays (grows) with time when $\alpha_{j,k}t + \beta_{j,k} > 0(<0)$. Using equations (21) and (22) in equation (2) yields the associated correlation amplitude profile given by

$$\langle\psi_j(t)|\psi_k(t)\rangle = \mathrm{e}^{-(\alpha_{j,k}t+\beta_{j,k})+\mathrm{i}\,(a_{j,k}t+b_{j,k})}. \tag{23}$$

Observe that (i) these results are in complete agreement with equations (15) and (16) for $\mathcal{L} = \hbar(a_{j,k} + i\alpha_{j,k})$; and (ii) equation (23) is a constraint upon the basis state probability amplitudes and time evolution operators for $|\psi_j\rangle$ and $|\psi_k\rangle$ which must be satisfied in order for $\mathcal{L}$ to be a constant of the motion. A similar development using time dependent functions instead of constants can obviously produce more complicated profiles. The application of these notions to correlation control and weak energy dynamics is discussed in [18].

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